THE LENGTH OF CHAINS IN ALGEBRAIC LATTICES

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ABSTRACT. We study how the existence in an algebraic lattice L of a chain of a given type is reflected in the join-semilattice $K(L)$ of its compact elements. We show that for every chain α of size κ, there is a set $\mathbb{B}$ of at most 2^{κ} join-semilattices, each one having a least element such that an algebraic lattice L contains no chain of order type $I(\alpha)$ if and only if the join-semilattice $K(L)$ of its compact elements contains no join-subsemilattice isomorphic to a member of $\mathbb{B}$. We show that among the join-subsemilattices of $[\omega]^{<\omega}$ belonging to $\mathbb{B}$, one is embeddable in all the others. We conjecture that if α is countable, there is a finite set $\mathbb{B}$.

INTRODUCTION

This paper is about the relationship between the length of chains in an algebraic lattice L and the structure of the join-semilattice $K(L)$ of the compact elements of L.

We started such an investigation in [2], [3], [4], [5]. Our motivation came from posets. Let P be an ordered set (poset). An *ideal* of P is any non-empty up-directed initial segment of P. The set $J(P)$ of ideals of P, ordered by inclusion, is an interesting poset associated with P, and it is natural to ask about the relationship between the two posets. For a concrete example, if $P := [\kappa]^{<\omega}$ the set, ordered by inclusion, consisting of finite subsets of a set of size κ, then $J([\kappa]^{<\omega})$ is isomorphic to $\mathfrak{P}(\kappa)$ the power set of κ ordered by inclusion. In [4] we proved:

Theorem 1. *A poset P contains a subset isomorphic to $[\kappa]^{<\omega}$ if and only if $J(P)$ contains a subset isomorphic to $\mathfrak{P}(\kappa)$.*

Maximal chains in $\mathfrak{P}(\kappa)$ are of the form $I(C)$, where $I(C)$ is the chain of initial segments of an arbitrary chain C of size κ (cf. [1]). Hence, if $J(P)$ contains a subset isomorphic to $\mathfrak{P}(\kappa)$ it contains a copy of $I(C)$ for every chain C of size κ, whereas chains in P can be small: eg in $P := [\kappa]^{<\omega}$ they are finite or have order type ω. What happens if for a given order type α, particularly a countable one, $J(P)$ contains no chain of type α? A partial answer was given by Pouzet, Zaguia, 1984 (cf. [11] Theorem 4, pp.62). In order to state their result, we recall that the order type α of a chain C is *indecomposable* if C is embeddable in I or in $C \setminus I$ for every initial segment I of C.

Theorem 2. [1] *Given an indecomposable countable order type α, there is a finite list of ordered sets $A_1^{\alpha}, A_2^{\alpha}, \ldots, A_{n_{\alpha}}^{\alpha}$ such that for every poset P, the set $J(P)$ of ideals of P*

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[1]In Theorem 4 [11], $I(\alpha)$ is replaced by α. This is due to the fact that if α is a countable indecomposable order type and P is a poset, $I(\alpha)$ can be embedded into $J(P)$ if and only if α can be embedded into $J(P)$.

contains no chain of type $I(\alpha)$ if and only if P contains no subset isomorphic to one of the $A_1^\alpha, A_2^\alpha, \ldots, A_{n_\alpha}^\alpha$.

Now, if P is a join-semilattice with a least element, $J(P)$ is an algebraic lattice, and moreover every algebraic lattice is isomorphic to the poset $J(K(L))$ of ideals of the join-semilattice $K(L)$ of the compact elements of L (see [9]). Due to the importance of algebraic lattices, it was natural to ask whether the two results above have an analog if posets are replaced by join-semilattices and subposets by join-subsemilattices. This question was the starting point of our research.

We immediately observed that the specialization of Theorem 1 to this case shows no difference. Indeed *a join-semilattice P contains a subset isomorphic to $[\kappa]^{<\omega}$ if and only if it contains a join-subsemilattice isomorphic to $[\kappa]^{<\omega}$.* Turning to the specialization of Theorem 2, we notice that it as to be quite different and is far from being immediate. In fact, we do not know yet whether for every countable α there is a finite list as in Theorem 2.

The purpose of this paper is to present the results obtained in that direction. In order to simplify the presentation, we will denote by $\mathbb{J}$ the class of join-semilattices having a least element. If $\mathbb{B} \subseteq \mathbb{J}$, we denote by $Forb(\mathbb{B})$ the class of $P \in \mathbb{J}$ which contain no join-subsemilattice isomorphic to a member of $\mathbb{B}$. If α denotes an order type, we denote by $\mathbb{J}_\alpha$ the class of members P of $\mathbb{J}$ such that the lattice $J(P)$ of ideals of P contains a chain of order type $I(\alpha)$. Finally we set $\mathbb{J}_{\neg\alpha} := \mathbb{J} \setminus \mathbb{J}_\alpha$.

Our first result expresses that a characterization as Theorem 2 is possible.

Theorem 3. *For every order type α there is a subset $\mathbb{B}$ of $\mathbb{J}$ of size at most $2^{|\alpha|}$ such that $\mathbb{J}_{\neg\alpha} = Forb(\mathbb{B})$.*

This is very weak. Indeed, we cannot answer the following question.

Question 1. *If α is countable, is there a finite $\mathbb{B}$?*

Examples are in order. Let ω be the order type of the chain $\mathbb{N}$ of non negative integers equipped with the natural order, ω^* be the order type of $\mathbb{N}$ equipped with the reverse order and η be the order type of the chain $\mathbb{Q}$ of rational numbers. Let $\underline{\Omega}(\omega^*)$ be the join-semilattice obtained by adding a least element to the set $[\omega]^2$ of two-element subsets of ω, identified to pairs (i, j), $i < j < \omega$, ordered so that $(i, j) \leq (i', j')$ if and only if $i' \leq i$ and $j \leq j'$ (see Figure 1).
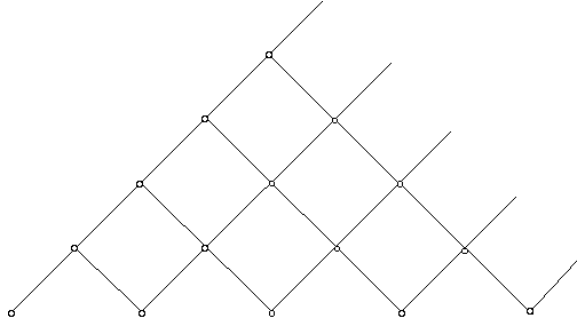

FIGURE 1. $\underline{\Omega}(\omega^*)$

Furthermore, let $\underline{\Omega}(\eta)$ be the poset represented Figure 2. It is immediate to see that if α is finite or ω, then the list in Theorem 2 has just one member, namely $A_1^\alpha = \{\alpha'\}$

(where α' is such that $\alpha = 1 + \alpha'$). In this case, the specialization to join-semilattice yields the same result. If $\alpha \in \{\omega^*, \eta\}$, it was proved in [11] that the list has two members: α and $\underline{\Omega}(\alpha)$. In this last case the specialization to join-semilattice is certainly different: the analogous list has at least three members, namely α, $[\omega]^{<\omega}$ and $\underline{\Omega}(\alpha)$. If $\alpha = \omega^*$, these three members suffice. If $\alpha = \eta$, we do not know. These very specific cases take into account important classes of posets. Let us say that a poset P is *well-founded*, resp. *scattered*, if it contains no chain of type ω^*, resp. η. We have:

Theorem 4. *(Theorem 1.3 [5]) An algebraic lattice L is well-founded if and only if $K(L)$ is well-founded and contains no join-subsemilattice isomorphic to $\underline{\Omega}(\omega^*)$ or to $[\omega]^{<\omega}$.*
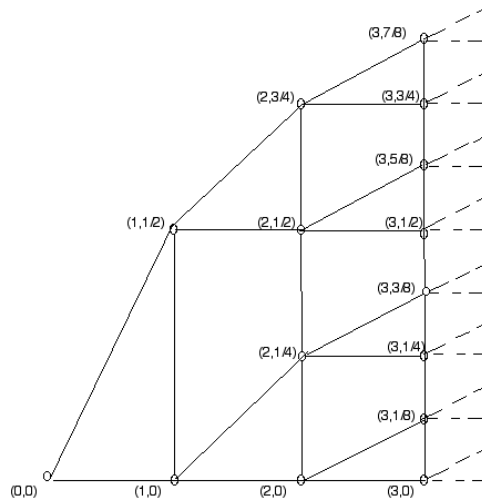

FIGURE 2. $\underline{\Omega}(\eta)$

Question 2. *Is it true that an algebraic lattice L is scattered if and only if $K(L)$ is scattered and contains no join-subsemilattice isomorphic to $\underline{\Omega}(\eta)$ or to $[\omega]^{<\omega}$?*

The join-semilattice $[\omega]^{<\omega}$ never appeared in the list mentionned in Theorem 2 but, as Theorem 4 illustrates, it might appear in the specialization to join-semilattices. This raises two questions. For which α it appears? If it does not appears, are they particular join-semilattices of $[\omega]^{<\omega}$ which appear? Here are the answers:

Theorem 5. *Let α be a countable order type.*
  (i) *The join-semilattice $[\omega]^{<\omega}$ belongs to every list $\mathbb{B}$ characterizing the class $\mathbb{J}_{\neg\alpha}$ of join-semilattices P such that $J(P)$ contains no chain of type $I(\alpha)$ if and only if α is not an ordinal.*
  (ii) *If α is an ordinal, then among the join-subsemilattices P of $[\omega]^{<\omega}$ which does not belong to $\mathbb{J}_{\neg\alpha}$ there is one, say Q_α, which embeds as a join-semilattice in all the others.*

The poset Q_α is of the form $I_{<\omega}(S_\alpha)$ where S_α is some sierpinskisation of α and ω. We recall that a *sierpinskisation* of a countable order type α and ω, or simply of α, is any poset $(S, \leq)$ such that the order on S is the intersection of two linear orders on S, one of type α, the other of type ω. Such a sierpinskisation can be obtained from a bijective map $\varphi : \omega \to \alpha$, setting $S := \mathbb{N}$ and $x \leq y$ if $x \leq y$ w.r.t. the natural order on $\mathbb{N}$ and

$\varphi(x) \leq \varphi(y)$ w.r.t. the order of type α. Let $\omega.\alpha$ be the ordinal sum of α copies of the chain ω; a sierpinskization of $\omega.\alpha$ and ω is *monotonic* if it obtained from a bijective map $\varphi : \omega \to \omega\alpha$ such that φ^{-1} is order-preserving on each subset of the form $\omega \times \{\beta\}$ where $\beta \in \alpha$. With that in hand, if α is an ordinal, we set $S_\alpha := \alpha$ if $\alpha < \omega$. If $\alpha = \omega\alpha' + n$ with $\alpha' \neq 0$ and $n < \omega$, let $S_\alpha := \Omega(\alpha') \oplus n$ be the direct sum of $\Omega(\alpha')$ and the chain n, where $\Omega(\alpha')$ is a monotonic sierpinskisation of $\omega\alpha'$ and ω. We note that for countably infinite α's, S_α is a sierpinskisation of α and ω.

Among the monotonic sierpinskisations of $\omega\alpha$ and ω there are some which are join-subsemilattices of the direct product $\omega \times \alpha$ that we call *lattice sierpinskisations*. Indeed, to every countable order type α, we may associate a join-subsemilattice $\Omega_L(\alpha)$ of the direct product $\omega \times \alpha$ obtained via a monotonic sierpinskisation of $\omega\alpha$ and ω. We add a least element, if there is none, and we denote by $\underline{\Omega}_L(\alpha)$ the resulting poset. Posets $\underline{\Omega}(\omega^*)$ and $\underline{\Omega}(\eta)$ represented Figure 1 and Figure 2 fall in this category. We also associate the join-semilattice P_α defined as follows:

If $1 + \alpha \not\leq \alpha$, in which case $\alpha = n + \alpha'$ with $n < \omega$ and α' without a first element, we set $P_\alpha := n + \underline{\Omega}_L(\alpha')$. If not, and if α is equimorphic to $\omega + \alpha'$ we set $P_\alpha := \underline{\Omega}_L(1 + \alpha')$, otherwise, we set $P_\alpha = \underline{\Omega}_L(\alpha)$.

The importance of this kind of lattice sierpinskization steems from the following result:

Theorem 6. *If α is countably infinite, P_α belongs to every list $\mathbb{B}$ characterizing $\mathbb{J}_{\neg\alpha}$.*

This work leaves open the following questions.

Questions 3.   (1) *If α is a countably infinite ordinal, does the minimal obstructions are α, P_α, Q_α and some lexicographical sums of obstructions corresponding to smaller ordinal?*

   (2) *If α is a scattered order type which is not an ordinal, does the minimal obstructions are α, P_α, $[\omega]^{<\omega}$ and some lexicographical sums of obstructions corresponding to smaller scattered order types?*

We only have some examples of ordinals for which the answer to the first question is positive. We conjecture that the answer is always positive.

To keep the paper at a reasonable length, we present only the proofs of Theorem 3 and Theorem 5. The proof of Theorem 6 and the presentation of examples supporting the conjecture are posponed to an other paper. The interested reader can find all the results mentionned here in [7] a paper available from the authors.

1. JOIN-SEMILATTICES AND A PROOF OF THEOREM 3

A *join-semilattice* is a poset P such that every two elements x, y have a least upper-bound, or join, denoted by $x \vee y$. If P has a least element, that we denote 0, this amounts to say that every finite subset of P has a join. In the sequel, we will mostly consider join-semilattices with a least element. Let Q and P be such join-semilattices. A map $f : Q \to P$ is *join-preserving* if:

$$(1) \qquad\qquad f(x \vee y) = f(x) \vee f(y)$$

for all $x, y \in Q$.

This map *preserves finite (resp. arbitrary) joins* if

$$(2) \qquad\qquad f\left(\bigvee X\right) = \bigvee\{f(x) : x \in X\}$$

for every finite (resp. arbitrary) subset X of Q.

If P is a join-semilattice with a least element, the set $J(P)$ of ideals of P ordered by inclusion is a complete lattice. If A is a subset of P, there is a least ideal containing A, that we denote $< A >$. An ideal I is *generated* by a subset A of P if $I = < A >$. If $[A]^{<\omega}$ denotes the collection of finite subsets of A we have:

$$(3) \qquad\qquad < A >= \downarrow \{\bigvee X : X \in [A]^{<\omega}\}$$

Lemma 1. *Let Q be a join-semilattice with a least element and L be a complete lattice. To a map $g : Q \to L$ associate $\overline{g} : \mathfrak{P}(Q) \to L$ defined by setting $\overline{g}(X) := \bigvee \{g(x) : x \in X\}$ for every $X \subseteq Q$. Then $\overline{g}$ induces a map from $J(Q)$ in L which preserves arbitrary joins whenever g preserves finite joins.*

Proof.

Claim 1. *Let $I \in J(Q)$. If g preserves finite joins and A generates I then $\overline{g}(I) = \bigvee \{g(x) : x \in A\}$.*

Proof of claim 1. Since $I = < A >$ and g preserves finite joins, $< \{g(x) : x \in I\} >= < \{g(x) : x \in A\} >$. The claimed equality follows. □

Now, let $\mathcal{I} \subseteq J(Q)$ and $I := \bigvee \mathcal{I}$. Clearly, $A := \bigcup \mathcal{I}$ generates I. Claim 1 yields $\overline{g}(I) = \bigvee \{g(x) : x \in A\} = \bigvee \bigcup \{\{g(x) : x \in J\} : J \in \mathcal{I}\} = \bigvee \{\bigvee \{g(x) : x \in J\} : J \in \mathcal{I}\} = \bigvee \{\overline{g}(J) : J \in \mathcal{I}\}$. This proves that $\overline{g}$ preserves arbitrary joins. □

Lemma 2. *Let R be a poset and P be a join-semilattice with a least element. The following properties are equivalent:*

(i) *There is an embedding from $I(R)$ in $J(P)$ which preserves arbitrary joins.*
(ii) *There is an embedding from $I(R)$ in $J(P)$.*
(iii) *There is a map g from $I_{<\omega}(R)$ in P such that*

$$(4) \qquad\qquad X \not\subseteq Y_1 \cup \ldots \cup Y_n \Rightarrow g(X) \not\leq g(Y_1) \vee \ldots \vee g(Y_n)$$

    for all $X, Y_1, \ldots, Y_n \in I_{<\omega}(R)$.
(iv) *There is a map $h : R \to P$ such that*

$$(5) \qquad\qquad \forall i (1 \leq i \leq n \Rightarrow x \not\leq y_i) \Rightarrow h(x) \not\leq h(y_1) \vee \ldots \vee h(y_n)$$

    for all $x, y_1, \ldots, y_n \in R$.

Proof. $(i) \Rightarrow (ii)$. Obvious.

$(ii) \Rightarrow (iii)$ Let f be an embedding from $I(R)$ in $J(P)$. Let $X \in I_{<\omega}(R)$. The set $A := Max(X)$ of maximal elements of X is finite and $X := \downarrow A$. Set $F(X) := \{f(R \setminus \uparrow a) : a \in Max(X)\}$ and $C(X) := f(X) \setminus \bigcup F(X)$.

Claim 2. $C(X) \neq \emptyset$ *for every $X \in I_{<\omega}(R)$*

Proof of Claim 2. If $X = \emptyset$, $F(X) = \emptyset$. Thus $C(X) = f(X)$ and our assertion is proved. We may then assume $X \neq \emptyset$. Suppose $C(X) = \emptyset$, that is $f(X) \subseteq \bigcup F(X)$. Since $f(X)$ is an ideal of P and $\bigcup F(X)$ is a finite union of initial segments of P, this implies that $f(X)$ is included in some, that is $f(X) \subseteq f(R \setminus \uparrow a)$ for some $a \in Max(X)$. Since f is an embedding, this implies $X \subseteq R \setminus \uparrow a$ hence $a \notin X$. A contradiction. □

Claim 2 allows us to pick an element $g(X) \in C(X)$ for each $X \in I_{<\omega}(R)$. Let g be the map defined by this process. We show that implication (4) holds. Let $X, Y_1, \ldots, Y_n \in I_{<\omega}(R)$. We have $g(Y_i) \in f(Y_i)$ for every $1 \leq i \leq n$. Since f is an embedding $f(Y_i) \subseteq f(Y_1 \cup \ldots \cup Y_n)$ for every $1 \leq i \leq n$. But $f(Y_1 \cup \ldots \cup Y_n)$ is an ideal. Hence $g(Y_1) \vee \ldots \vee g(Y_n) \in f(Y_1 \cup \ldots \cup Y_n)$. Suppose $X \not\subseteq Y_1 \cup \ldots \cup Y_n$. There is $a \in Max(X)$ such that $Y_1 \cup \ldots \cup Y_n \subseteq R \setminus \uparrow a$. And since f is an embedding, $f(Y_1 \cup \ldots \cup Y_n) \subseteq f(R \setminus \uparrow a)$.

If $g(X) \leq g(Y_1) \vee \ldots \vee g(Y_n)$ then $g(X) \in f(Y_1 \cup \ldots \cup Y_n)$. Hence $g(X) \in f(R \backslash \uparrow a)$, contradicting $g(X) \in C(X)$.

$(iii) \Rightarrow (iv)$. Let $g : I_{<\omega}(R) \rightarrow P$ such that implication (4) holds. Let h be the map induced by g on R by setting $h(x) := g(\downarrow x)$ for $x \in R$. Let $x, y_1, \ldots, y_n \in R$. If $x \not\leq y_i$ for every $1 \leq i \leq n$, then $\downarrow x \not\subseteq (\downarrow y_1 \cup \ldots \cup \downarrow y_n)$. Since g satisfies implication (4), we have $h(x) := g(\downarrow x) \not\leq g(\downarrow y_1) \vee \ldots \vee g(\downarrow y_n) = h(y_1) \vee \ldots \vee h(y_n)$. Hence implication (5) holds.

$(iv) \Rightarrow (i)$ Let $h : R \rightarrow P$ such that implication (5) holds. Define $f : I(R) \rightarrow J(P)$ by setting $f(I) :=< \{h(x) : x \in I\} >$, the ideal generated by $\{h(x) : x \in I\}$, for $I \in I(R)$. Since in $I(R)$ the join is the union, f preserves arbitrary joins. We claim that f is one-to-one. Let $I, J \in I(R)$ such that $I \not\subseteq J$. Let $x \in I \setminus J$. Clearly $h(x) \in f(I)$. We claim that $h(x) \notin f(J)$. Indeed, if $h(x) \in f(J)$, then $h(x) \leq \bigvee\{h(y) : y \in F\}$ for some finite subset F of J. Since implication (5) holds, we have $x \leq y$ for some $y \in F$. Hence $x \in J$, contradiction. Consequently $f(I) \not\subseteq f(J)$. Thus f is one-to-one as claimed.   □

   The following proposition rassembles the main properties of the comparizon of join-semilattices.

Proposition 1. *Let P, Q be two join-semilattices with a least element. Then:*
   (1) Q is embeddable in P by a join-preserving map iff Q is embeddable in P by a map preserving finite joins.
   (2) If Q is embeddable in P by a join-preserving map then $J(Q)$ is embeddable in $J(P)$ by a map preserving arbitrary joins.
       Suppose $Q := I_{<\omega}(R)$ for some poset R. Then:
   (3) Q is embeddable in P as a poset iff Q is embeddable in P by a map preserving finite joins.
   (4) $J(Q)$ is embeddable in $J(P)$ as a poset iff $J(Q)$ is embeddable in $J(P)$ by a map preserving arbitrary joins.
   (5) If $\downarrow x$ is finite for every $x \in R$ then Q is embeddable in P as a poset iff $J(Q)$ is embeddable in $J(P)$ as a poset.

Proof.

(1) Let $f : Q \rightarrow P$ satisfying $f(x \vee y) = f(x) \vee f(y)$ for all $x, y \in Q$. Set $g(x) := f(x)$ if $x \neq 0$ and $g(0) := 0$. Then g preserves finite joins.

(2) Let $f : Q \rightarrow P$ and $\overline{f} : J(Q) \rightarrow J(P)$ defined by $f(I) :=\downarrow \{f(x) : x \in I\}$. If f preserves finite joins, then $\overline{f}$ preserves arbitrary joins. Furthermore, $\overline{f}$ is one to one provided that f is one-to-one.

(3) Let $f : Q \rightarrow P$. Taking account that $Q := I_{<\omega}(R)$, set $g(\emptyset) := 0$ and $g(I) := \bigvee\{f(\downarrow x) : x \in I\}$ for each $I \in I_{<\omega}(R) \setminus \{\emptyset\}$. Since in Q the join is the union, the map g preserves finite unions.

(4) This is equivalence $(i) \Longleftrightarrow (iv)$ of Lemma 2.

(5) If Q is embeddable in P as a poset, then from Item (3), Q is embeddable in P by a map preserving finite joins. Hence from Item (1), $J(Q)$ is embeddable in $J(P)$ by a map preserving arbitrary joins. Conversely, suppose that $J(Q)$ is embeddable in $J(P)$ as a poset. Since $J(Q)$ is isomorphic to $I(R)$, Lemma 2 implies that there is map h from R in P such that implication (5) holds. According to the proof of Lemma 2, the map $f : I(R) \rightarrow J(P)$ defined by setting $f(I) := \bigvee\{h(x) : x \in I\}$ is an embedding preserving arbitrary joins. Since $\downarrow x$ is finite for every $x \in R$, I is finite , hence $f(I)$ has a largest element, for every $I \in I_{<\omega}(R)$. Thus f induces an embedding from Q in P preserving finite joins.

□

Theorem 7. *Let R be a poset, $Q := I_{<\omega}(R)$ and $\kappa := |Q|$. Then, there is a set $\mathbb{B}$, of size at most 2^κ, made of join-semilattices, such that for every join-semilattice P, the join-semilattice $J(Q)$ is not embeddable in $J(P)$ by a map preserving arbitrary joins if and only if no member Q_f of $\mathbb{B}$ is embeddable in P as a join-semilattice.*

Proof. If $\downarrow x$ is finite for every $x \in R$ the conclusion of the theorem holds with $\mathbb{B} = \{Q\}$ (apply Item (3), (4), (5) of Proposition 1). So we may assume that R is infinite. Let P be a join-semilattice. Suppose that there is an embedding f from $J(Q)$ in $J(P)$ which preserves arbitrary joins.

Claim 3. *There is a join-semilattice Q_f such that*

    (1) Q_f embeds in P as a join-semilattice.
    (2) $J(Q)$ embeds in $J(Q_f)$ by a map preserving arbitrary joins.
    (3) $|Q_f| = |Q|$.

Proof of Claim 3. From Lemma 2, there is a map $g : Q \to P$ such that inequality (4) holds. Let Q_f be the join-semilattice of P generated by $\{g(x) : x \in Q\}$. This inequality holds when P is replaced by Q_f. Thus from Proposition 1, $J(Q)$ embeds into $J(Q_f)$ by a map preserving arbitrary joins. Since R is infinite, $|Q_f| = |Q| = |R|$. □

For each join-semilattice P and each embedding $f : J(Q) \to J(P)$ select Q_f, given by Claim 3, on a fixed set of size κ. Let $\mathbb{B}$ be the collection of this join-semilattices. Since the number of join-semilattices on a set of size κ is at most 2^κ, $|\mathbb{B}| \le 2^\kappa$. □

Proof of Theorem 3. Let α be the order type of a chain C. Apply Theorem 7 above with $R := C$. Since, in this case, $J(Q)$ is embeddable in $J(P)$ if and only if $J(Q)$ is embeddable in $J(P)$ by a map preserving arbitrary joins, Theorem 3 follows. □

2. Sierpinskisations and a proof of Theorem 5

2.1. **Proof of item (i) of Theorem 5.** We start with the following lemma

Lemma 3. *If α is a countably infinite order type and S is a sierpinskisation of α and ω then the join-semilattice $I_{<\omega}(S)$, made of finitely generated initial segments of S, is isomorphic to a join-subsemilattice of $[\omega]^{<\omega}$ and belongs to $\mathbb{J}_\alpha$.*

Proof. By definition, the order on a sierpinskisation S of α and ω has a linear extension such that the resulting chain $\overline{S}$ has order type α. Hence, from a result of [1], the chain $I(\overline{S})$ is a maximal chain of $I(S)$ of type $I(\alpha)$. The lattices $I(S)$ and $J(I_{<\omega}(S))$ are isomorphic, thus $I_{<\omega}(S) \in \mathbb{J}_\alpha$. The order on S has a linear extension of type ω, thus every principal initial segment of S is finite and more generally every finitely generated initial segment of S is finite. This tells us that $I_{<\omega}(S)$ is a join-subsemilattice of $[S]^{<\omega}$. Since S is countable, $I_{<\omega}(S)$ identifies to a join-subsemilattice of $[\omega]^{<\omega}$. □

The proof of the "only if" part goes as follows. Let S be a sierpinskization of α and ω. According to Lemma 3, the join-semilattice $I_{<\omega}(S)$, made of finitely generated initial segments of S, is isomorphic to a join-subsemilattice of $[\omega]^{<\omega}$ and belongs to $\mathbb{J}_\alpha$. If $[\omega]^{<\omega}$ is a minimal member of $\mathbb{J}_\alpha$, then $[\omega]^{<\omega}$ is embeddedable as a join-subsemilattice in $I_{<\omega}(S)$. To conclude that α cannot be an ordinal, it suffices to prove:

Lemma 4. *If α is an ordinal and S is a sierpinskisation of α and ω, then $[\omega]^{<\omega}$ is not embeddable in $I_{<\omega}(S)$.*

This simple fact relies on the important notion of well-quasi-ordering introduced by Higman [10]. We recall that a poset P is *well-quasi-ordered* (briefly w.q.o.) if every non-empty subset A of P has at least a minimal element and the number of these minimal elements is finite. As shown by Higman, this is equivalent to the fact that $I(P)$ is well-founded [10].

Well-ordered set are trivially w.q.o. and, as it is well known, the direct product of finitely many w.q.o. is w.q.o. Lemma 4 follows immediately from this. Indeed, if S is a sierpinskisation of α and ω, it embeds in the direct product $\omega \times \alpha$. Thus S is w.q.o. and consequently $I(S)$ is well-founded. This implies that $[\omega]^{<\omega}$ is not embeddable in $I_{<\omega}(S)$. Otherwise $J([\omega]^{<\omega})$ would be embeddable in $J(I_{<\omega}(S))$, that is $\mathfrak{P}(\omega)$ would be embeddable in $I(S)$. Since $\mathfrak{P}(\omega)$ is not well-founded, this would contradict the well-foundedness of $I(S)$.

The "if" part is based on our earlier work on well-founded algebraic lattices, and essentially on the following corollary of Theorem 4.

Theorem 8. *(Corollary 1.1, [5]) A join-subsemilattice P of $[\omega]^{<\omega}$ contains either $[\omega]^{<\omega}$ as a join-semilattice or is well-quasi-ordered. In the latter case, $J(P)$ is well-founded.*

With this result, the proof of the "if" part of Theorem 5 is immediate. Indeed, suppose that α is not an ordinal. Let $P \in \mathbb{J}_\alpha$. The lattice $J(P)$ contains a chain isomorphic to $I(\alpha)$. Since α is not an ordinal, $\omega^* \leq \alpha$. Hence, $J(P)$ is not well-founded. If P is embeddable in $[\omega]^{<\omega}$ as a join-semilattice then, from Theorem 8, P contains a join-subsemilattice isomorphic to $[\omega]^{<\omega}$. Thus $[\omega]^{<\omega}$ is minimal in $\mathbb{J}_\alpha$.

A sierpinskisation S of a countable order type α and ω is embeddable into $[\omega]^{<\omega}$ as a poset. A consequence of Theorem 8 is the following

Corollary 1. *If S can be embedded in $[\omega]^{<\omega}$ as a join-semilattice, α must be an ordinal.*

Proof. Otherwise, S contains an infinite antichain and by Theorem 8 it contains a copy of $[\omega]^{<\omega}$. But this poset cannot be embedded in a sierpinskisation. Indeed, a sierpinskisation is embeddable into a product of two chains, whereas $[\omega]^{<\omega}$ cannot be embedded in a product of finitely many chains (for every integer n, it contains the power set $\mathfrak{P}(\{0, \ldots, n-1\})$ which cannot be embedded into a product of less than n chains; its dimension, in the sense of Dushnik-Miller's notion of dimension, is infinite, see [12]). $\quad\square$

2.2. **Proof of item (ii) of Theorem 5.** We prove first that there is a sierpinskisation S of α and ω such that $Q := I_{<\omega}(S) \in \mathbb{J}_\alpha$ is embeddable in P by a map preserving finite joins.

Theorem 9. *Let α be a countable ordinal and $P \in \mathbb{J}_\alpha$. If P is embeddable in $[\omega]^{<\omega}$ by a map preserving finite joins there is a sierpinskisation S of α and ω such that $I_{<\omega}(S) \in \mathbb{J}_\alpha$ and $I_{<\omega}(S)$ is embeddable in P by a map preserving finite joins.*

Proof. We construct first R such that $I_{<\omega}(R) \in \mathbb{J}_\alpha$ and $I_{<\omega}(R)$ is embeddable in P by a map preserving finite joins.

We may suppose that P is a subset of $[\omega]^{<\omega}$ closed under finite unions. Thus $J(P)$ identifies with the set of arbitrary unions of members of P. Let $(I_\beta)_{\beta<\alpha+1}$ be a strictly increasing sequence of ideals of P. For each $\beta < \alpha$ pick $x_\beta \in I_{\beta+1} \setminus I_\beta$ and $F_\beta \in P$ such that $x_\beta \in F_\beta \subseteq I_{\beta+1}$. Set $X := \{x_\beta : \beta < \alpha\}$, $\rho := \{(x_{\beta'}, x_{\beta''}) : \beta' < \beta'' < \alpha$ and $x_{\beta'} \in F_{\beta''}\}$. Let $\hat{\rho}$ be the reflexive transitive closure of ρ. Since $\theta := \{(x_{\beta'}, x_{\beta''}) : \beta' < \beta'' < \alpha\}$ is a linear order containing ρ, $\hat{\rho}$ is an order on X. Let $R := (X, \hat{\rho})$ be the resulting poset.

Claim 4. $I_{<\omega}(R) \in \mathbb{J}_\alpha$.

Proof of claim 4. The linear order θ extends the order $\hat{\rho}$ and has type α, thus $I(R)$ has a maximal chain of type $I(\alpha)$. Since $J(I_{<\omega}(R))$ is isomorphic to $I(R)$, $I_{<\omega}(R)$ belongs to $\mathbb{J}_\alpha$ as claimed. □

Claim 5. *For each $x \in X$, the initial segment $\downarrow x$ in R is finite.*

Proof of claim 5. Suppose not. Let β be minimum such that for $x := x_\beta$, $\downarrow x$ is infinite. For each $y \in X$ with $y < x$ in R select a finite sequence $(z_i(y))_{i \leq n_y}$ such that:

(1) $z_0(y) = x$ and $z_{n_y} = y$.
(2) $(z_{i+1(y)}, z_i(y)) \in \rho$ for all $i < n_y$.

According to item 2, $z_1(y) \in F_\beta$. Since F_β is finite, it contains some $x' := x_{\beta'}$ such that $z_1(y) = x'$ for infinitely many y. These elements belong to $\downarrow x'$. The fact that $\beta' < \beta$ contradicts the choice of x. □

Claim 6. *Let ϕ be defined by $\phi(I) := \bigcup\{F_\beta : x_\beta \in I\}$ for each $I \subseteq X$. Then:*
ϕ induces an embedding of $I(R)$ in $J(P)$ and an embedding of $I_{<\omega}(R)$ in P.

Proof of claim 6. We prove the first part of the claim. Clearly, $\phi(I) \in J(P)$ for each $I \subseteq X$. And trivially, ϕ preserves arbitrary unions. In particular, ϕ is order preserving. Its remains to show that ϕ is one-to-one. For that, let $I, J \in I(R)$ such that $\phi(I) = \phi(J)$. Suppose $J \not\subseteq I$. Let $x_\beta \in J \setminus I$, Since $x_\beta \in J$, $x_\beta \in F_\beta \subseteq \phi(J)$. Since $\phi(J) = \phi(I)$, $x_\beta \in \phi(I)$. Hence $x_\beta \in F_{\beta'}$ for some $\beta' \in I$. If $\beta' < \beta$ then since $F_{\beta'} \subseteq I_{\beta'+1} \subseteq I_\beta$ and $x_\beta \notin I_\beta$, $x_\beta \notin F_{\beta'}$. A contradiction. On the other hand, if $\beta < \beta'$ then, since $x_\beta \in F_{\beta'}$, $(x_\beta, x_{\beta'}) \in \rho$. Since I is an initial segment of R, $x_\beta \in I$. A contradiction too. Consequently $J \subseteq I$. Exchanging the roles of I and J, yields $I \subseteq J$. The equality $I = J$ follows. For the second part of the claim, it suffices to show that $\phi(I) \in P$ for every $I \in I_{<\omega}(R)$. This fact is a straightforward consequence of Claim 5. Indeed, from this claim I is finite. Hence $\phi(I)$ is finite and thus belongs to P.

Claim 7. *The order $\hat{\rho}$ has a linear extension of type ω.*

Proof of claim 7. Clearly, $[\omega]^{<\omega}$ has a linear extension of type ω. Since R embeds in $[\omega]^{<\omega}$, via an embedding in P, the induced linear extension on R has order type ω. □
Let ρ' be the intersection of such a linear extension with the order θ and let $S := (X, \rho')$.

Claim 8. *For every $I \in I(S)$, resp. $I \in I_{<\omega}(S)$ we have $I \in I(R)$, resp. $I \in I_{<\omega}(R)$.*

Proof of claim 8. The first part of the proof follows directly from the fact that ρ' is a linear extension of $\hat{\rho}$. The second part follows from the fact that each $I \in I_{<\omega}(S)$ is finite. □
It is then easy to check that the poset S satisfies the properties stated in the theorem. □

In order to conclude, it suffices to prove that one can replace Q by $Q_\alpha := I_{<\omega}(S_\alpha)$, where S_α is the sierpinskization defined in the introduction.
This fact follows directly from Lemma 7 below. It relies on properties of monotonic sierpinskizations, some already in [11].
We recall that for a countable order type α', two monotonic sierpinskisations of $\omega\alpha'$ and ω are embeddable in each other and denoted by the same symbol $\Omega(\alpha')$ and we recall the following result (cf. [11] Proposition 3.4.6. pp. 168).

Lemma 5. *Let α' be a countable order type. Then $\Omega(\alpha')$ is embeddable in every sierpinskisation S' of $\omega\alpha'$ and ω.*

Lemma 6. *Let α be a countably infinite order type and S be a sierpinskisation of α and ω. Assume that $\alpha = \omega\alpha' + n$ where $n < \omega$. Then there is a subset of S which is the direct sum $S' \oplus F$ of a sierpinskisation S' of $\omega\alpha'$ and ω with an n-element poset F.*

Proof.   Assume that S is given by a bijective map φ from $\mathbb{N}$ onto a chain C having order type α. Let A' be the set of the n last elements of C, $A := \varphi^{-1}(A')$ and a be the largest element of A in $\mathbb{N}$. The image of $]a \rightarrow)$ has order type $\omega\alpha'$, thus S induces on $]a \rightarrow)$ a sierpinskisation S' of $\omega\alpha'$ and ω. Let F be the poset induced by S on A. Since every element of S' is incomparable to every element of F these two posets form a direct sum.   □

Let α be a countably infinite order type such that $\alpha = \omega\alpha' + n$ where $n < \omega$. We set $S_\alpha := \Omega(\alpha') \oplus n$ and $Q_\alpha := I_{<\omega}(S_\alpha)$.

Lemma 7. *$Q_\alpha \in \mathbb{J}_\alpha$ and for every sierpinskisation S of α and ω, Q_α is embeddable in $I_{<\omega}(S)$ by a map preserving finite joins.*

Proof.   For the the first part, apply Lemma 3.

Case 1. $n = 0$. By Lemma 5, $\Omega(\alpha')$ is embeddable in S. Thus Q_α is embeddable in $I_{<\omega}(S)$ by a map preserving finite joins.

Case 2. $n \neq 0$. Apply Lemma 6. According to Case 1, $I_{<\omega}(\Omega(\alpha'))$ is embeddable in $I_{<\omega}(S')$. On an other hand $n + 1$ is embeddable in $I_{<\omega}(F) = I(F)$. Thus Q_α which is isomorphic to the product $I_{<\omega}(\Omega(\alpha')) \times (n + 1)$ is embeddable in the product $I_{<\omega}(S') \times I_{<\omega}(F)$. This product is itself isomorphic to $I_{<\omega}(S' \oplus F)$. Since $S' \oplus F$ is embeddable in S, $I_{<\omega}(S' \oplus F)$ is embeddable in $I_{<\omega}(S)$ by a map preserving finite joins. It follows that Q_α is embeddable in $I_{<\omega}(S)$ by a map preserving finite joins.   □

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